CS4311 Design and Analysis of Algorithms

Homework 4

Due: 11:10 am, June 12, 2008 (before class)

1. Let $G = (V, E)$ be a connected, undirected graph. An *articulation point* of G is a vertex whose removal will disconnect G.

   Suppose we perform DFS on G, and let T be the resulting DFS tree. We are going to find all articulation points of G based on T.

   (a) (25%) Prove that the root of T is an articulation point of G if and only if it has at least two children in T.

   Hint: ($\Leftarrow$) If the root of T has two children, c_1 and c_2. Can they be connected by a path in G without the root? (White-path theorem may be useful.)

   (b) (25%) Let v be a non-root vertex of T. Prove that v is an articulation point of G if and only if v has a child s such that there is no back edge from s or any descendant of s to a proper ancestor of v.

   Hint: ($\Rightarrow$) If for each child s of v, there is a back edge from s or its descendant to a proper ancestor of v, can you show that every neighbor of v is connected to parent(v)? (Be careful!!! Some neighbors of v may not be children of v.) In this case, can v be an articulation point?

   Hint: ($\Leftarrow$) If all back edges from s or from its descendant do not point to any proper ancestor of v, where can they point to? Can you show that if v is removed, s and parent(v) are disconnected?

   (c) (25%) Let

   $$low[v] = \min \begin{cases} d(v), \\ d(w) & \text{such that } (u, w) \text{ is a back edge from some descendant } u \text{ of } v. \end{cases}$$

   Show how to compute $low[v]$ for all vertices $v \in V$ in $O(|E|)$ time.

   (Hint: Recall T is the DFS tree. Suppose for a node v, its children are $c_1, c_2, \ldots, c_k$. Can you show any relationship among $low[v]$ and $low[c_1], low[c_2], \ldots, low[c_k]$?)

   (d) (25%) Show how to compute all articulation points in $O(|E|)$ time.

2. (Bonus: 10%)[§] A directed graph is said to be *semi-connected* if for all pairs of vertices u and v, we have $u \rightsquigarrow v$, or $v \rightsquigarrow u$, or both. (The notation $u \rightsquigarrow v$ means u can reach v by a directed path.)

   (a) (5%) Suppose G is a directed acyclic graph with n vertices, and suppose we have performed a topological sort on G. Let v_i denote the ith vertex in the topological sort order.

   Show that G is semi-connected if and only if there is an edge (v_i, v_{i+1}) for all $i = 1, 2, \ldots, n-1$.

   (b) (5%) Suppose G is a general directed graph (which may contains cycle). Give an $O(|V|+|E|)$-time algorithm to check if G is semi-connected. Show that your algorithm is correct.

   (Hint: Finding SCC, then topological sort on component graph.)

[§] Q2 is a bonus question. Total mark is calculated by: Q1 $\times$ (100%+ Q2).