CS5371 Theory of Computation

Homework 4
Due: 3:20 pm, December 15, 2006 (before class)

1. Let $\Gamma = \{\texttt{0}, \texttt{1}, \sqcup\}$ be the tape alphabet of all TMs in this problem. Define the ***busy beaver function*** $BB : \mathbb{N} \to \mathbb{N}$ as follows. For each value of k, consider all k-state TMs that halt when started with a blank tape. Let $BB(k)$ be the maximum number of $\texttt{1}$s that remain on the tape among all of these machines. Show that BB is not a computable function.

2. Let $AMBIG_{\text{CFG}} = \{\langle G \rangle \mid G \text{ is an ambiguous CFG}\}$. Show that $AMBIG_{\text{CFG}}$ is undecidable. (Hint: Use a reduction from PCP. Given an instance

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right] \right\},$$

of the Post Correspondence Problem, construct a CFG G with the rules

$$\begin{aligned} &S \to T \mid B \\ &T \to t_1 T \texttt{a}_1 \mid \cdots \mid t_k T \texttt{a}_k \mid t_1 \texttt{a}_1 \mid \cdots \mid t_k \texttt{a}_k \\ &B \to b_1 B \texttt{a}_1 \mid \cdots \mid b_k B \texttt{a}_k \mid b_1 \texttt{a}_1 \mid \cdots \mid b_k \texttt{a}_k, \end{aligned}$$

where $\texttt{a}_1, \ldots, \texttt{a}_k$ are new terminal symbols. Prove that this reduction work.)

3. Define a ***two-headed finite automaton***(2DFA) to be a deterministic finite automaton that has two *read-only*, bidirectional heads that start at the left-hand end of the input tape and can be independently controlled to move in either direction. The tape of a 2DFA is finite and is just large enough to contain the input plus two additional blank tape cells, one on the left-hand end and one on the right-hand end, that serve as delimiters. A 2DFA accepts its input by entering a special accept state. For example, a 2DFA can recognize the language $\{\texttt{a}^n\texttt{b}^n\texttt{c}^n \mid n \geq 0\}$.

   (a) Let $A_{\text{2DFA}} = \{\langle M, x \rangle \mid M \text{ is a 2DFA and } M \text{ accepts } x\}$. Show that A_{2DFA} is decidable.

   (b) Let $E_{\text{2DFA}} = \{\langle M \rangle \mid M \text{ is a 2DFA and } L(M) = \{\}\}$. Show that E_{2DFA} is not decidable.

4. Let $J = \{w \mid \text{either } w = \texttt{0}x \text{ for some } x \in A_{TM}, \text{ or } w = \texttt{1}y \text{ for some } y \notin A_{TM}\}$. Show that neither J nor the complement of J is Turing-recognizable.

5. **Rice's theorem.** Let P be any nontrivial property of the language of a Turing machine. Prove that the problem of determining whether a given Turing machine's language has property P is undecidable.

   In more formal terms, let P be a language consisting of Turing machine descriptions where P fulfills two conditions. First, P is nontrivial—it contains some, but not all, TM descriptions. Second, P is a property of the TM's language—whenever $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in P$ if and only if $\langle M_2 \rangle \in P$. Here, M_1 and M_2 are any TMs. Prove that P is an undecidable language.