CS 2336
Discrete Mathematics

Lecture 18

Trees: Spanning Trees

Outline

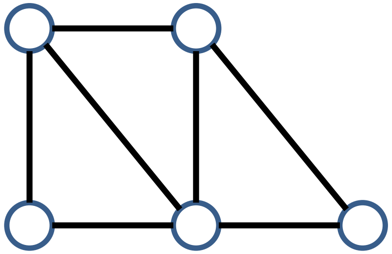
- What is a Spanning Tree ?
- Counting Spanning Trees
- Prüfer Code

What is a Spanning Tree ?

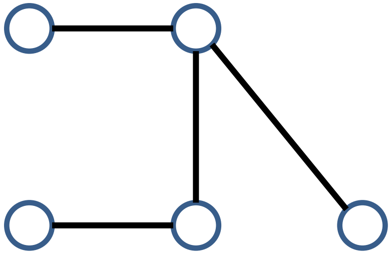
- Let G be a simple graph

> Definition: A spanning tree of G is a subgraph of G that is a tree containing every vertex of G

- Ex :



G



a spanning tree of G

What is a Spanning Tree ?

- Let G be a simple graph

Theorem :

G is connected ⇔ G has a spanning tree

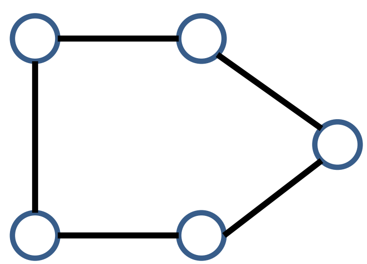
- Proof :

The "if" case is trivial. For the "only if" case, we repeatedly remove an edge from a loop in G until G contains no loop. The resulting graph must be connected, and thus is a spanning tree of G.

Counting Spanning Trees

- How many spanning trees are there in the following graphs ?

1



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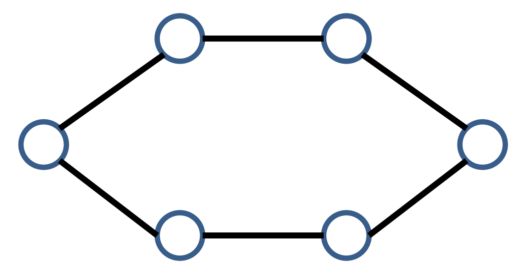

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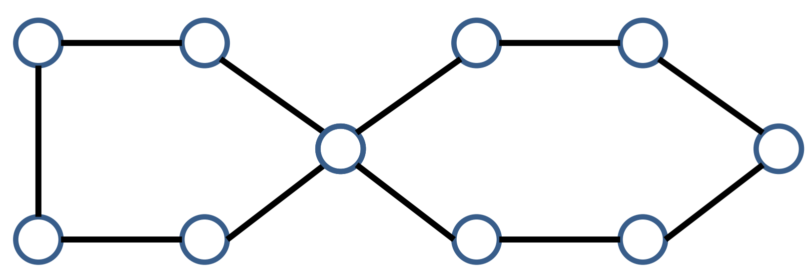
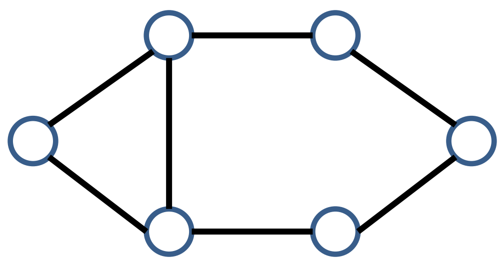


Counting Spanning Trees

- How many spanning trees are there in the following graphs ?

4



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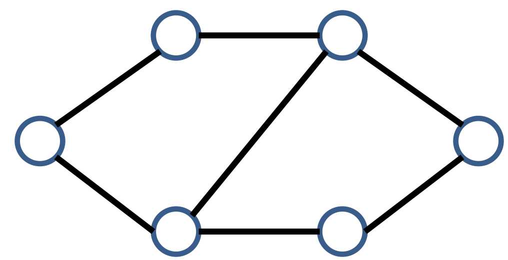

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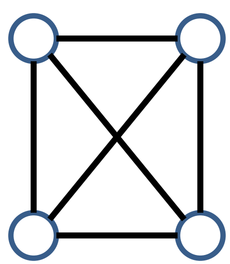

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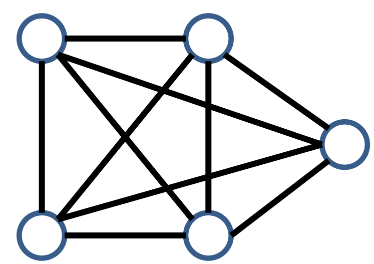


Counting Spanning Trees

- Consider the complete graph K_n , with $n \geq 2$

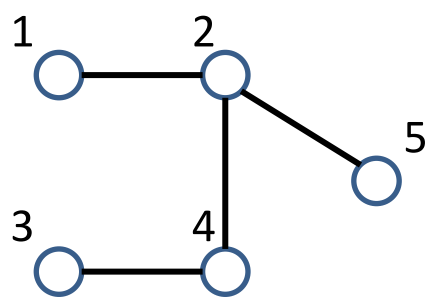
Theorem :

K_n has n^{n-2} different spanning trees

- The above is known as the Cayley's formula
- The theorem can be proven by induction, through the use of Prüfer codes (see next page)

Prüfer Codes

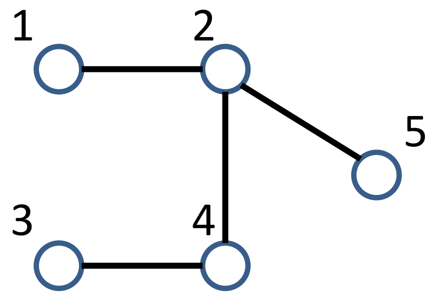
- Suppose that vertices of K_n are labeled by 1 to n
- Consider a spanning tree of K_n



- We recursively remove the leaf with the smallest label, and write down its neighbor

➔ Stop when there are only two leaves

Prüfer Codes



- The sequence for the above tree is :

  2 (remove 1), 4 (remove 3), 2 (remove 4)

- It is easy to check that for a different spanning tree of K_n, the sequence will be different

  ➔ This sequence is called the Prufer code

Prüfer Codes

- On the other hand, given any sequence of length n – 2, with each entry from 1 to n

  ➔ there is a unique spanning tree !

- Ex : Sequence = 2, 4, 2
- Here, the smallest missing number is 1, and the first in the sequence is 2, so we know that :

  1 is a leaf, and its neighbor is 2

Prüfer Codes

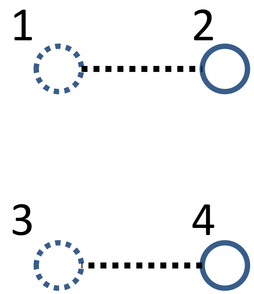
- We also know that 1 will be removed next in the construction of the remaining sequence

➔ the remaining sequence is 4, 2, and now the smallest missing number is 3, so we know that :

3 is a leaf, and its neighbor is 4

in the remaining tree

Prüfer Codes



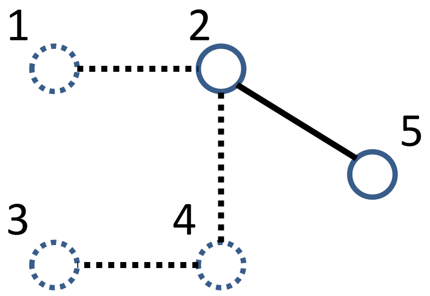
- Next, 3 will be removed in the construction of the remaining sequence

➔ the remaining sequence is 2, and now the smallest missing number is 4, so we know that :

4 is a leaf, and its neighbor is 2

in the remaining tree

Prüfer Codes



- Next, 4 will be removed in the construction of the remaining sequence

  → as the remaining leaves are 2 and 5

  they become neighbors in the remaining tree

  → we obtain the original tree