# CS5314 Randomized Algorithms

Homework 4
Due: 3:20 pm, June 12, 2008 (before class)

1. (20%) Let $S$ be a set of elements, and $C$ be a collection of subsets, $S_1, S_2, S_3, ..., S_{|C|}$, such that $|S_i| = k \geq 2$ for each $i$. Suppose the number of subsets, $|C|$, is at most $4^{k-1} - 1$.[1]

   Show that we can color the elements of $S$ with 4 colors such that no $S_i$ is monochromatic.

2. (20%) Let $F$ be a family of subsets of $N = \{1, 2, ..., n\}$. $F$ is called an *anti-chain* if there are no $A, B \in F$ satisfying $A \subset B$. Let $\sigma$ be a random permutation of the elements of $N$.

   Let $X$ be the random variable with

   $$X = |\{i : \{\sigma(1), \sigma(2), ...\sigma(i)\} \in F\}|.$$

   By considering the expectation of $X$ prove that $|F| \leq \binom{n}{\lfloor n/2 \rfloor}$.

   (Hint: (i) What are the possible values of $X$ ? (ii) $F$ can be partitioned into $n$ parts according to the size of its subsets. Let $k_i$ denotes the number of size-$i$ subsets, so that $|F| = k_1 + k_2 + ... + k_n$. Can you relate $\mathrm{E}[X]$ with $k_1, k_2, \ldots, k_n$ ?)

3. (20%) A *tournament* is a graph with exactly one directed edge between each pair of vertices. Show that there is a tournament $T$ with $n$ vertices such that $T$ contains at least $n!2^{-(n-1)}$ Hamiltonian paths.

4. (20%) Consider a random graph in $G_{n,p}$, with $p = 1/n$. Let $X$ be the number of triangles in the graph, where a triangle is a clique with three edges. Show that

   $$\Pr(X \geq 1) \leq 1/6$$

   and that

   $$\lim_{n\to\infty} \Pr(X \geq 1) \geq 1/7$$

   (Hint: For the second part, use the conditional expectation inequality.)

5. (20%) Use the general form of the Lovasz local lemma to prove that the symmetric version can be improved by replacing the condition $4dp \leq 1$ by the weaker condition $ep(d+1) \leq 1$.

6. (Bonus: 10%) A set $S$ is *sum-free* if for any $s_i$ and $s_j$ (not necessarily distinct) in $S$, their sum $s_i + s_j$ is not in $S$.

   For example, $\{1, 3, 5, 9\}$ is sum-free, but $\{1, 2, 3\}$, $\{2, 4\}$ are not.

   Let $B = \{b_1, b_2, ..., b_n\}$ be a set of $n$ positive integers, with $b_1 < b_2 < \cdots < b_n$. In this question, we want to show that $B$ must contain a sum-free subset with size at least $n/3$.

   (a) Let $p = 3k+2$ be a prime such that $p > b_n$. Create a set $C = \{k+1, k+2, ..., 2k+1\}$. Show that $C$ is sum-free, even under modulo-$p$ arithmetic. (That is, for any $c_i$ and $c_j$ (not necessarily distinct) in $C$, their sum $c_i + c_j \pmod{p} \notin C$.)

[1] In graph theory, a hypergraph is a graph where edges can connect any number of vertices. In our problem, $(S, C)$ is in fact a hypergraph, where $S$ is the set of vertices and $C$ is the set of edges.

(b) Suppose we randomly select $x$ with $x \in [1, p-1]$, and calculate $d_1, d_2, ..., d_n$ with $d_i = xb_i \pmod{p}$. Show that

$$\Pr(d_i \in C) > \frac{|C|}{p-1} > \frac{1}{3}.$$

Hint: If $x \neq x'$, can $xb_i = x'b_i$ under modulo-$p$ arithmetic? Hence, conclude that under modulo-$p$ arithmetic,

$$\{b_i, 2b_i, \ldots, (p-1)b_i\} = \{1, 2, 3, \ldots, p-1\}.$$

(c) Argue that there exists some $x^*$ in part (b) such that at least $n/3$ of the corresponding $d_i$'s are in $C$.

(d) Let $A$ be the set of $b_i$'s such that $x^* b_i \in C$. Show that $A$ is sum-free.