CS5314
Randomized Algorithms

Lecture 24: Markov Chains
(Parrondo's Paradox)

Objectives

- Introduce Parrondo's Paradox
  - named after a Spanish physicist Juan Parrondo (1964--)
- The Paradox describes an interesting example of two games A and B, such that if we play any one of them (say A) in the long run, we will be losing, but …

  if each time, we choose A or B to play with equal probability, then we may be winning in the long run !!!

Game A

- Game A is very simple: We have a biased coin, such that

  it comes up head with probability 0.49,
  it comes up tail with probability 0.51

- In the game, you will win $1 if head comes up, and lose $1 otherwise…

Question: If you can play Game A again and again, would you like to do so?

Game B

- Game B is a bit complicated: We have two biased coins. Depending on the current money you have, we will choose which of the biased coins to use

1st coin: (when your money not multiple of 3)

it comes up head with probability 0.74,
it comes up tail with probability 0.26

2nd coin: (when your money is multiple of 3)

it comes up head with probability 0.09,
it comes up tail with probability 0.91

Game B (cont)

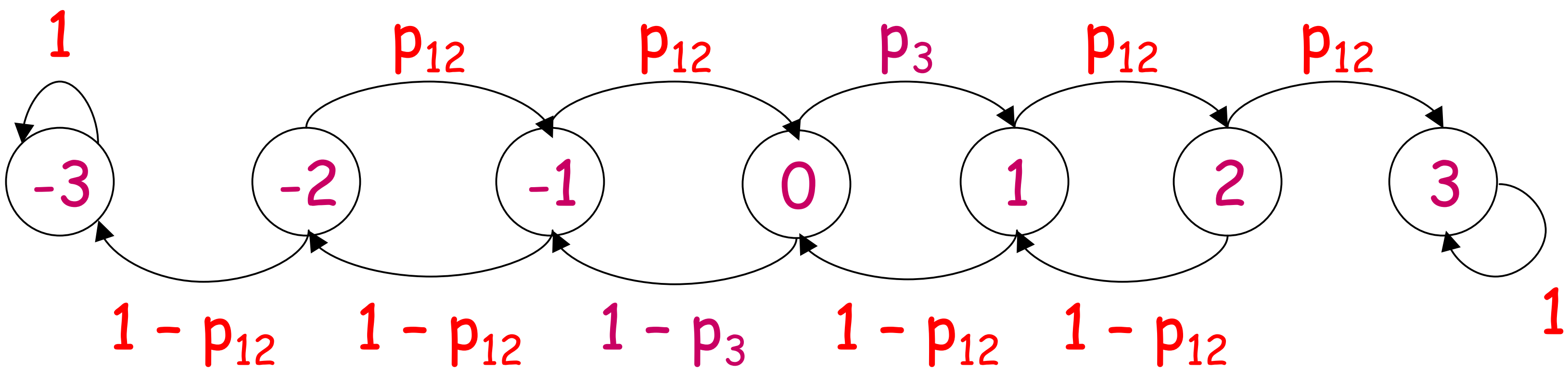
- Again, in this game, you will win \$1 if head comes up, and lose \$1 otherwise
- In general, Game B can be stated as: we win with probability p_{12} if our money is not a multiple of 3, and we win with probability p_3 otherwise

Question: If you can play Game B again and again, would you like to do so?

Idea. We play Game B only if it is more likely to win \$3 before losing \$3 … [why?]

Markov Chain for Game B

- The previous idea can be modeled by the following Markov chain:



Markov Chain for Game B (2)

- Let z_j be the probability of winning \$3 before losing \$3 when starting at state j
- Based on this definition, we have:

$$z_{-3} = 0 \text{ and } z_3 = 1$$

Also, we have:

$$z_{-2} = (1-p_{12})\, z_{-3} + p_{12}\, z_{-1}$$
$$z_{-1} = (1-p_{12})\, z_{-2} + p_{12}\, z_0$$
$$z_0 = (1-p_3)\, z_{-1} + p_3\, z_1$$
$$z_1 = (1-p_{12})\, z_0 + p_{12}\, z_2$$
$$z_2 = (1-p_{12})\, z_1 + p_{12}\, z_3$$

Markov Chain for Game B (3)

- Since p_{12} and p_3 are given, the previous system has 7 equations and 7 unknowns, so that it can be solved easily
- In particular, we have:

$$z_0 = p_3 p_{12}^2 \,/\, ((1-p_3)(1-p_{12})^2 + p_3 p_{12}^2)$$

- By definition, z_0 = prob of winning \$3 before losing \$3, when starting with \$0

  ➔ if $z_0 > 0.5$, we should play Game B again and again; else, we should not …

Markov Chain for Game B (4)

In our example,

$$p_{12} = 0.74 \text{ and } p_3 = 0.09$$

Thus,

$$z_0 = p_3 p_{12}^2 \,/\, \left((1-p_3)(1-p_{12})^2 + p_3 p_{12}^2 \right)$$

$$= (0.09)\,(0.74)^2 \,/$$

$$\left((0.91)(0.26)^2 + (0.09)(0.74)^2 \right)$$

$$= 0.049284 \,/\, 0.1108$$

$$< 0.5$$

So, Game B is also a bad choice for us …

Game A + Game B

- After going through the above analysis, we know that neither Game A nor Game B is a good choice to play in the long run …

- Now, we have Game C as follows:
  1. Flip a fair coin.
  2. If head comes up, we play Game A
     Else, we play Game B

- That means, after a game, we will still either win $1 or lose $1

Game C

Question: If you can play Game C again and again, would you like to do so?

Intuition: Roughly speaking, if we play Game C again and again, we will play Game A and play Game B each 50% of the time…

Both A and B are not favorable for us …

So, it seems like Game C is losing …

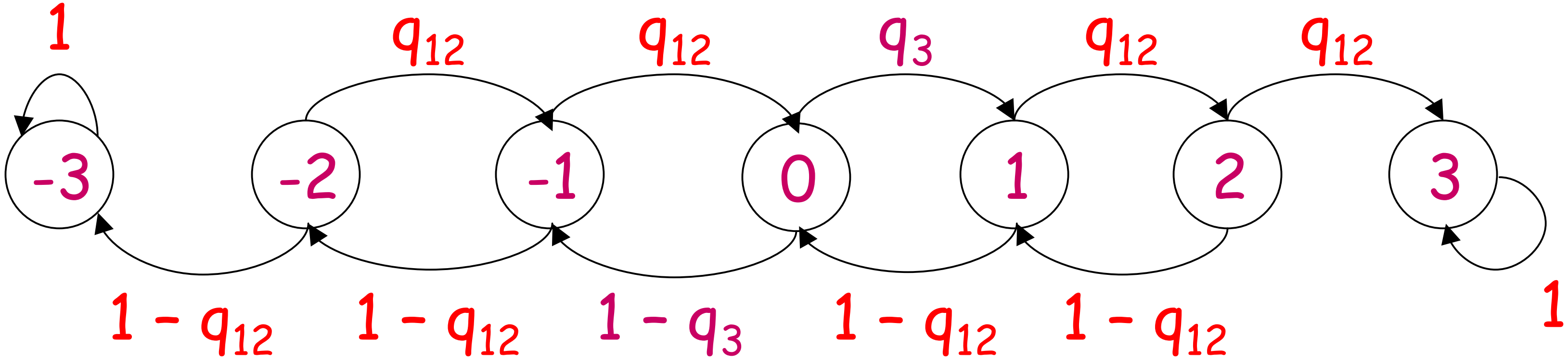
But, is it really true??

Game C (cont)

- Let us analyze whether we should play Game C using the same idea as before
- First, let q_{12} be the probability of winning when our money is not a multiple of 3, and let q_3 be the probability of winning when our money is a multiple of 3
- Again, we play Game C only if it is more likely to win \$3 before losing \$3

Markov Chain for Game C

- The previous idea can be modeled by the following Markov chain:



Markov Chain for Game C (2)

- Thus, the probability of winning \$3 in Game C before losing \$3, when we start with \$0, is:

$$z = q_3 q_{12}^2 \,/\, \left((1-q_3)(1-q_{12})^2 + q_3 q_{12}^2 \right)$$

Question: What are the values of q_{12} and q_3 in our example??

Ans. $q_{12} = 0.5 * 0.49 + 0.5 * 0.74 = 0.615$

$q_3 = 0.5 * 0.49 + 0.5 * 0.09 = 0.29$

Markov Chain for Game C (3)

Thus,

$$z = q_3 q_{12}^2 \,/\, \left((1-q_3)(1-q_{12})^2 + q_3 q_{12}^2 \right)$$

$$= (0.29)\,(0.615)^2 \,/$$

$$\left((0.71)(0.385)^2 + (0.29)(0.615)^2 \right)$$

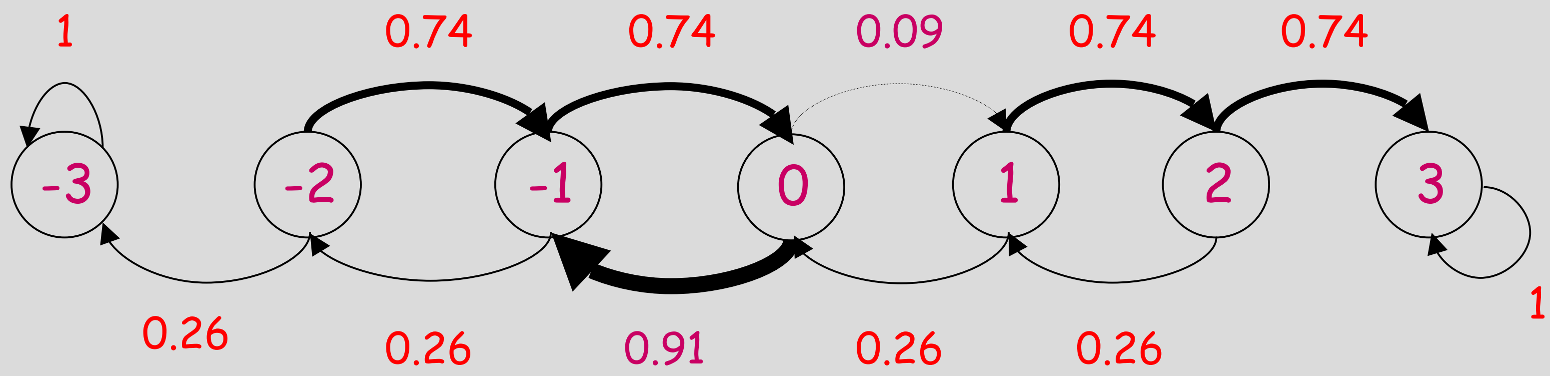
$$= 0.10968525 \,/\, 0.214925$$

$$> 0.5$$

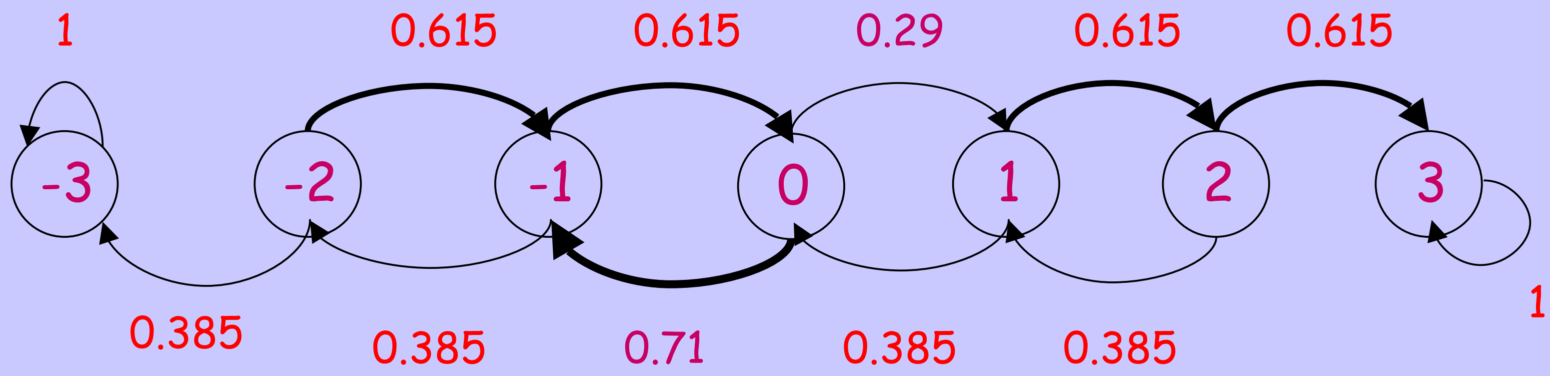
So, Game *C* is a good choice for us !!!

How does Game A help us?

This is Game B



This is Game C



Final Remarks

I hope you like and enjoy the course

(... Sorry that I haven't enough time to cover all the interesting topics in the textbook

➔ I hope that you can find your spare time to read the uncovered chapters...)

Thanks Joyce for being a wonderful tutor

Thanks all of you for coming to the class!

Finally, Good Luck! in the exam ^_^