Applications of Fourier Expansion

(1) Show that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

(Solution) Let $x^2 = \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k \cos(k\pi x) + b_k \sin(k\pi x)]$ for $x \in [-1, 1]$.

Take the integration on both sides, we have $a_0 = \int_{-1}^{1} x^2 dx = \frac{2}{3}$.

Next, we know that x^2 is an even function and $\{\sin(k\pi x),\ k = 1, 2, \cdots\}$ are *odd* functions, thus, $b_k = 0,\ \forall\ k = 1, 2, \cdots$.

Furthermore, $\{1, \cos(\pi x), \sin(\pi x), \cos(2\pi x), \sin(2\pi x), \cdots\}$ is an orthogonal set of functions on $[-1, 1]$.

Then $a_m = \int_{-1}^{1} x^2 \cos(m\pi x)dx = \frac{4\times(-1)^m}{m^2\pi^2},\ for\ m = 1, 2, \cdots$. Hence, we have

$$x^2 = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{4 \times (-1)^k}{k^2\pi^2} \cos(k\pi x).$$

If we plug in with $x = 1$, then we have

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \qquad (1)$$

Definition For $x \in (-1,1)$, define the Chebyshev polynomial of degree n by

$$T_n(x) = cos(ncos^{-1}x) \ \forall \ n \geq 0$$

Denote the inner product by $\langle T_n, T_m \rangle = \int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}dx$

(a) Write down $T_n(x)$ in the polynomial format for $0 \leq n \leq 4$.

(Sa) Denote $cos^{-1}x = \theta$, then

$$\begin{aligned} T_{n+1}(x) &= cos((n+1)cos^{-1}x) &= cos((n+1)\theta), \\ T_n(x) &= cos(ncos^{-1}x) &= cos(n\theta), \\ T_{n-1}(x) &= cos((n-1)cos^{-1}x) &= cos((n-1)\theta). \end{aligned}$$

Then,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad \forall \ n = 0, 1, 2, \ \cdots \tag{2}$$

Thus, $T_0(x) = 1, \ T_1(x) = x, \ T_2(x) = 2x^2 - 1, \ T_3(x) = 4x^3 - 3x, \ T_4(x) = 8x^4 - 8x^2 + 1.$

(b) Show that $\langle T_n, T_m \rangle = 0$ if $n \neq m$.

(Sb)

$$\begin{aligned} \langle T_n, T_m \rangle &= \int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}dx \\ &= \int_0^{\pi} [cos(n\theta)cos(m\theta)]d\theta \\ &= 0 \ \ if \ \ n \neq m \end{aligned}$$

(c) Compute $\langle T_0, T_0 \rangle$, and $\langle T_n, T_n \rangle$ for $n \geq 1$.

(Sc) For $n \geq 1$,

$$\begin{aligned} \langle T_n, T_n \rangle &= \int_{-1}^{1} T_n(x)T_n(x)\frac{1}{\sqrt{1-x^2}}dx \\ &= \int_0^{\pi} [cos(n\theta)cos(n\theta)]d\theta \\ &= \int_0^{\pi} cos^2(n\theta)d\theta \\ &= \int_0^{\pi} \frac{1+cos(2n\theta}{2}d\theta \\ &= \frac{\pi}{2} \ \ if \ \ n \geq 1 \end{aligned}$$

Similarly, $\langle T_0, T_0 \rangle = \pi$.

(d) Find all of the roots of $T_n(x) = 0$.

(Sd) $cos(\frac{\pi}{n}(k + \frac{1}{2})), \ n \in N, \ k \in Z$