Exam for Scientific Computing, 2020

14:20-15:20, December 9, 2020

Name:____________ **StuID:**_____________ **IndexN:**_____

1.(10%) Approximate $ln(1.5)$ by using Taylor expansion for $f(x) = ln(1 + x)$ with $x \in (-1, 1)$ about $x = 0$ such that the accuracy is within 10^{-4}.

2.(10%) Let the matrix B and the vector $\mathbf{x}$ be defined as follows.

$$B = \begin{bmatrix} 3 & 4 & -7 & 1 \\ 5 & -7 & 10 & 3 \\ 2 & -5 & -9 & -4 \\ -8 & 10 & -8 & -5 \end{bmatrix}, \quad \mathbf{x} = [-4, 3, -2, -1]^t.$$

(a) Find $\|\mathbf{x}\|_{\mathbf{1}}$ and $\|\mathbf{x}\|_\infty$, respectively.

(b) Find $\|B\|_1$ and $\|B\|_\infty$, respectively.

3.(10%) Write down the matrix H obtained from the following Matlab code.

```
format rat
H=hilb(4)
```

4.(20%) Give the following Matlab code.

```
A=[1, 2, 3; 2, 6, 10; 3, 14, 28];
b=[1; 0; -8];
format short
X=A\b
```

(a) Find the *LU-decomposition* for A.

(b) What does the above Matlab code do?

(c) What is the output of X?

5.(20%) Let a matrix A be defined as

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -2 & 1 & -1 \\ 4 & -5 & 0 \end{bmatrix}$$

After an implementation of the $LU-Decomposition$ algorithm given below. on the input matrix A. Give the contents of output A.

♡ *LU-Decomposition Algorithm*

for $i = 1, 2, \cdots, n-1$

for $k = i+1, i+2, \cdots, n$

$m_{ki} \leftarrow a_{ki}/a_{ii}$ if $a_{ii} \neq 0$

$a_{ki} \leftarrow m_{ki}$

for $j = i+1, i+2, \cdots, n$

$a_{kj} \leftarrow a_{kj} - m_{ki} * a_{ij}$

endfor

endfor

endfor

6.(10%) Suppose that a tridiagonal matrix $T \in R^{n \times n}$ is also diagonally dominant.

(a) Give an efficient algorithm to do $T = LU$.

(b) How many floating-point operations (flops) are needed for your algorithm?

7.(10%) Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

Find the linear least squares solution for $A\mathbf{x} = \mathbf{b}$.

8.(10%) Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be eigenvalues of $A \in R^{n \times n}$, show that the trace of matrix A^k is the sum of $\{\lambda_i^k,\ i = 1, 2, \cdots, n\}$, that is, $tr(A^k) = \sum_{i=1}^{n} \lambda_i^k$.

Exam for Scientific Computing, 2020

Due by 15:50, December 16, 2020

Name:____________ **StuID:**_____________ **IndexN:**______

1.(10%) Apply $LU-decomposition$ with partial pivoting and back substitution to solve the following linear system of equations such that the accuracy is within 10^{-4}.

$$\begin{array}{ccccccc} x & + & \frac{1}{2}y & + & \frac{1}{3}z & = & 1 \\ \frac{1}{2}x & + & \frac{1}{3}y & + & \frac{1}{4}z & = & 1 \\ \frac{1}{3}x & + & \frac{1}{4}y & + & \frac{1}{5}z & = & 1 \end{array}$$

2.(20%) Find the characteristic polynomial of the matrix A given below and compare the roots of the characteristic equation of A with those obtained from the matlab command $eig(A)$.

$$A = \begin{bmatrix} -7 & 3 & 1 \\ 3 & -7 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

3.(10%) Give the nonlinear system of equations with variables $x = X(1), y = X(2), z = X(3)$ that the following Matlab codes attempt to solve.

```
n=3;
X=[1; 1; 1];
Nrun=10;
for k=1:Nrun
  F=[X(1)*X(2)-X(3)^2-1; ...
  X(1)*X(2)*X(3)-X(1)^2+X(2)^2-2; ...
  exp(X(1))-exp(X(2))-X(3)-3];
  A=[X(2),             X(1),             -2*X(3); ...
  X(2)*X(3)-2*X(1), X(1)*X(3)+2*X(2), X(1)*X(3); ...
  exp(X(1)),        exp(X(2)),        -1];
  s=0.002;  % Perturbation to avoid the near singularity
  dX=(A+s*eye(n))\F;
  X=X-dX;
end
format short
[X'; F']    % Output solution and error (ideally F==0)
```

Answer the following questions.

4.(20%) For $x \in (-1,1)$, define the Chebyshev polynomial of degree n by

$$T_n(x) = cos(ncos^{-1}x) \ \forall \ n \geq 0$$

Denote the inner product by $\langle T_n, T_m \rangle = \int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}dx$

(a) Write down $T_n(x)$ in the polynomial format for $0 \leq n \leq 4$.

(b) Show that $\langle T_n, T_m \rangle = 0$ if $n \neq m$.

(c) Compute $\langle T_0, T_0 \rangle$, and $\langle T_n, T_n \rangle$ for $n \geq 1$.

(d) Find all of the roots of $T_n(x) = 0$.

5.(20%) Using the nodes $x_0 = 2,\ x_1 = 2.5,\ and\ x_2 = 4$ to find a quadratic interpolating polynomial $P_2(x)$ for $f(x) = \frac{1}{x}$.

(a) Find the Lagrange polynomials $L_{2,0}(x), L_{2,1}(x),\ and\ L_{2,2}(x)$.

(b) Write $P_2(x)$ in form of Lagrange polynomials.

(c) Write $P_2(x)$ in the form of Newton divided difference.

(d) If $P_2(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0$, what are $\alpha_2, \alpha_1, \alpha_0$, respectively?

6.(20%) A natural cubic spline S on [0,2] is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & if\ 0 \leq x < 1 \\ S_1(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3, & if\ 1 \leq x \leq 2 \end{cases}$$

Find $a, b, c,\ and\ d$, respectively.