# Solutions for P3+P4 of Exam, 2020

*Due by 15:50, December 16, 2020*

**1.(10%)** Apply $LU-decomposition$ with partial pivoting and back substitution to solve the following linear system of equations such that the accuracy is within $10^{-4}$.

$$
\begin{array}{ccccccc}
x & + & \frac{1}{2}y & + & \frac{1}{3}z & = & 1 \\
\frac{1}{2}x & + & \frac{1}{3}y & + & \frac{1}{4}z & = & 1 \\
\frac{1}{3}x & + & \frac{1}{4}y & + & \frac{1}{5}z & = & 1
\end{array}
$$

**Ans:** $[3.0000;\ -24.0000;\ 30.0000]$

**2.(20%)** Find the characteristic polynomial of the matrix $A$ given below and compare the roots of the characteristic equation of $A$ with those obtained from the matlab command $eig(A)$.

$$
A = \begin{bmatrix} -7 & 3 & 1 \\ 3 & -7 & 2 \\ 1 & 2 & 5 \end{bmatrix}
$$

**Ans:** $P(x) = x^3 + 9x^2 - 35x - 247$, $[-10.0350;\ -4.4707;\ 5.5056]$

**3.(10%)** The nonlinear system of equations with variables $x = X(1), y = X(2), z = X(3)$ to be solved is

$$
\begin{array}{ccccccc}
xy & - & z^2 & & & = & 1 \\
xyz & - & x^2 & + & y^2 & = & 2 \\
e^x & - & e^y & - & Z & = & 3
\end{array}
$$

**Ans:** $[2.1659;\ 1.4493;\ 1.4625]$

**4.** For $x \in (-1, 1)$, define the Chebyshev polynomial of degree $n$ by

$$T_n(x) = cos(ncos^{-1}x) \ \forall \ n \geq 0$$

Denote the inner product by $\langle T_n, T_m \rangle = \int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}dx$

**(a)** $T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x,\ T_4(x) = 8x^4 - 8x^2 + 1.$

**(b)** Refer to class lectures.

**(c)** $\langle T_0, T_0 \rangle = \pi$ and $\langle T_n, T_n \rangle = \frac{\pi}{2}$ for $n \geq 1$.

**(d)** $cos(\frac{\pi}{n}(k + \frac{1}{2})),\ n \in N,\ k \in Z$

**5.**

**(b)** $0.5 \cdot \frac{(x-2.5)(x-4)}{(2-2.5)(2-4)} + 0.4 \cdot \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} + 0.25 \cdot \frac{(x-2)(x-2.5)}{(4-2)(4-2.5)}$

**(c)** $0.5 - 0.2 \cdot (x - 2) + 0.05 \cdot (x - 2)(x - 2.5)$

**(d)** $0.05 \cdot x^2 - 0.425 \cdot x + 1.15$

**6.(20%)** A natural cubic spline $S$ on [0,2] is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & if\ 0 \leq x < 1 \\ S_1(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3, & if\ 1 \leq x \leq 2 \end{cases}$$

Find $a, b, c,\ and\ d$, respectively.

**(Answer)** $(a, b, c, d) = (2, -1, -3, 1)$