Proof of the Central Limit Theorem

Theorem: Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $N(\mu, \sigma^2)$. Define $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$, then $\overline{X} \sim N(\mu, \sigma^2/n)$.

Theorem: Let $\overline{X}$ be the mean of a random sample $X_1, X_2, \ldots, X_n$ of size n from a distribution with mean μ and variance σ^2. Define $W_n = (\overline{X} - \mu)/(\sigma/\sqrt{n})$. Then

(a) $W_n = (\sum_{i=1}^n X_i - n\mu)/(\sqrt{n}\sigma)$

(b) $P(W_n \le w) \approx \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \Phi(w)$.

(c) $W_n \sim N(0,1)$ as $n \to \infty$.

(Proof)

$$\begin{aligned} E[\exp(tW_n)] &= E\left\{\exp\left[\left(\tfrac{t}{\sqrt{n}\sigma}\right)\left(\textstyle\sum_{i=1}^n X_i - n\mu\right)\right]\right\} \\ &= E\left\{\exp\left[\left(\tfrac{t}{\sqrt{n}}\right)\left(\tfrac{X_1-\mu}{\sigma}\right) + \cdots + \left(\tfrac{t}{\sqrt{n}}\right)\left(\tfrac{X_n-\mu}{\sigma}\right)\right]\right\} \\ &= E\left\{\exp\left[\left(\tfrac{t}{\sqrt{n}}\right)\left(\tfrac{X_1-\mu}{\sigma}\right)\right]\right\} \cdots E\left\{\exp\left[\left(\tfrac{t}{\sqrt{n}}\right)\left(\tfrac{X_n-\mu}{\sigma}\right)\right]\right\}, \end{aligned}$$

which follows from the independence of $X_1, X_2, \cdots, X_n$. Then

$$E[\exp(tW_n)] = \left[M\left(\tfrac{t}{\sqrt{n}}\right)\right]^n, \quad -h < \tfrac{t}{\sqrt{n}} < h,$$

where

$$M(t) = E\left\{\exp\left[t\left(\tfrac{X_i-\mu}{\sigma}\right)\right]\right\}, \quad -h < t < h$$

is the common moment-generating function of each

$$Y_i = \frac{X_i - \mu}{\sigma}, \quad i = 1, 2, \cdots, n.$$

since $E(Y_i) = 0$ and $E(Y_i^2) = 1$, it must be that

$$M(0) = 1, \quad M'(0) = E\left(\frac{X_i-\mu}{\sigma}\right) = 0, \quad M''(0) = E\left[\left(\frac{X_i-\mu}{\sigma}\right)^2\right] = 1$$

Hence, using Taylor's formula with a remainder, we know that there exists a number t_1 between 0 and t such that

$$M(t) = M(0) + M'(0)t + \frac{M''(t_1)t^2}{2} = 1 + \frac{M''(t_1)t^2}{2}.$$

Adding and subtracting $t^2/2$, we have

$$M(t) = 1 + \frac{t^2}{2} + \frac{]M''(t_1) - 1]t^2}{2}.$$

Using this expression of $M(t)$ in $E[\exp(tW_n)]$, we can represent the moment-generating function of W_n by

$$\begin{aligned} E[\exp(tW_n)] &= \left\{1 + \tfrac{1}{2}\left(\tfrac{t}{\sqrt{n}}\right)^2 + \tfrac{1}{2}[M''(t_1) - 1]\left(\tfrac{t}{\sqrt{n}}\right)^2\right\}^n \\ &= \left\{1 + \tfrac{t^2}{2n} + \tfrac{[M''(t_1)-1]t^2}{2n}\right\}^n, \quad -\sqrt{n}h < t < \sqrt{n}h, \end{aligned}$$

where now t_1 is between 0 and $t/\sqrt{n}$. Since $M''(t)$ is continuous at $t = 0$ and $t_1 \to 0$ as $n \to \infty$, we have

$$lim_{n\to\infty}[M''(t_1) - 1] = 1 - 1 = 0$$

Thus,

$$\begin{aligned} lim_{n\to\infty}E[\exp(tW_n)] &= lim_{n\to\infty}\left\{1 + \tfrac{t^2}{2n} + \tfrac{[M''(t_1)-1]t^2}{2n}\right\}^n \\ &= lim_{n\to\infty}\left\{1 + \tfrac{t^2}{2n}\right\}^n = e^{t^2/2} \end{aligned}$$

for all real t. We know that $e^{t^2/2}$ is the moment-generating function of the standard normal distribution, $N(0,1)$. Therefore, the limiting distribution of

$$W_n = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma} = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n\sigma^2}} \longrightarrow N(0,1) \quad as \quad n \to \infty.$$