6. and 7. Continuous Random Variables and Special Continuous Distributions

- Definition 6.1 on P.243:

Let X be a random variable. Suppose that there exists a nonnegative real-values function $f : R \to [0, \infty)$ such that for any subset of real numbers A that can be constructed from intervals by a countable number of set operations,

$$P(X \in A) = \int_A f(x)dx.$$

Then X is called absolutely continuous, or for simplicity, continuous. The f is called the *probability density function* (p.d.f., or pdf) of X. *F*(*t*)=$\int_{-\infty}^{t} f(x)dx$ is called the (cumulative) *distribution function* (c.d.f., or cdf) of X.

6.1 Probability Density Function

6.2 Density Function of a Function of a Random Variable

6.3 Expectations and Variances

6. Continuous Random Variables

- Properties of p.d.f. (pdf) and c.d.f. (cdf):

The probability density function (p.d.f.) *f and its distribution function F* of a continuous random variable X has the following properties.

(a) $F(t) = \int_{-\infty}^{t} f(x)\, dx$. $A = (-\infty, t]$, $F(t) = P(X \le t) = P(X \in A)$.

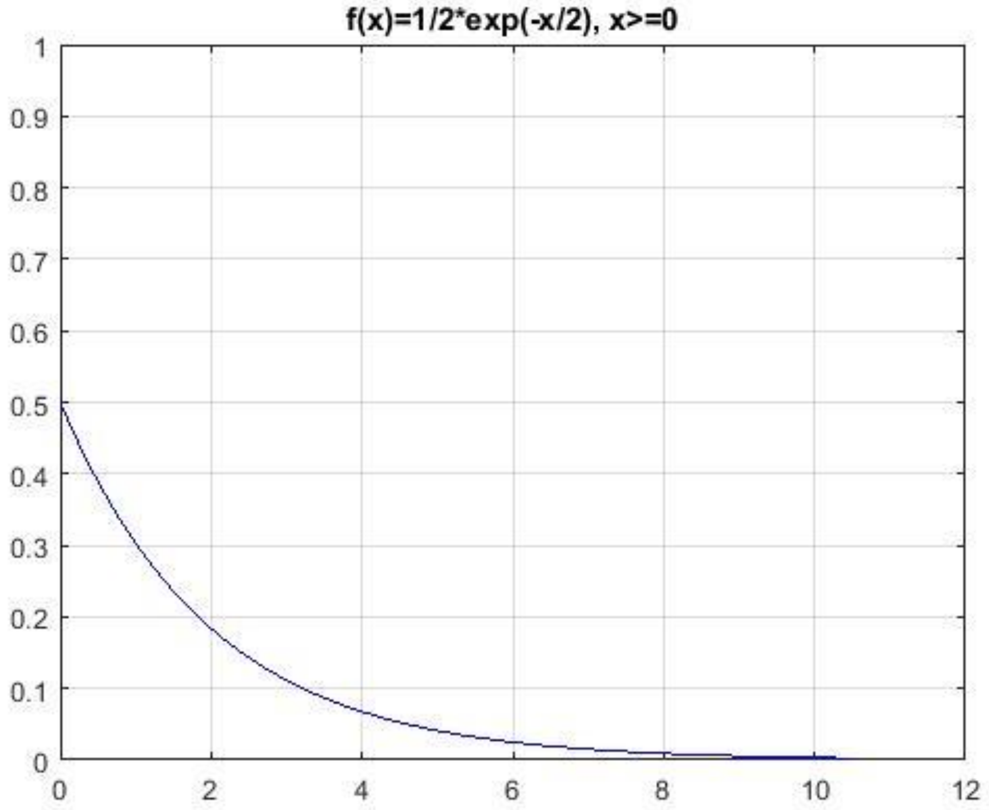
(b) $P(X \in R) = \int_{-\infty}^{\infty} f(x) dx = 1$.

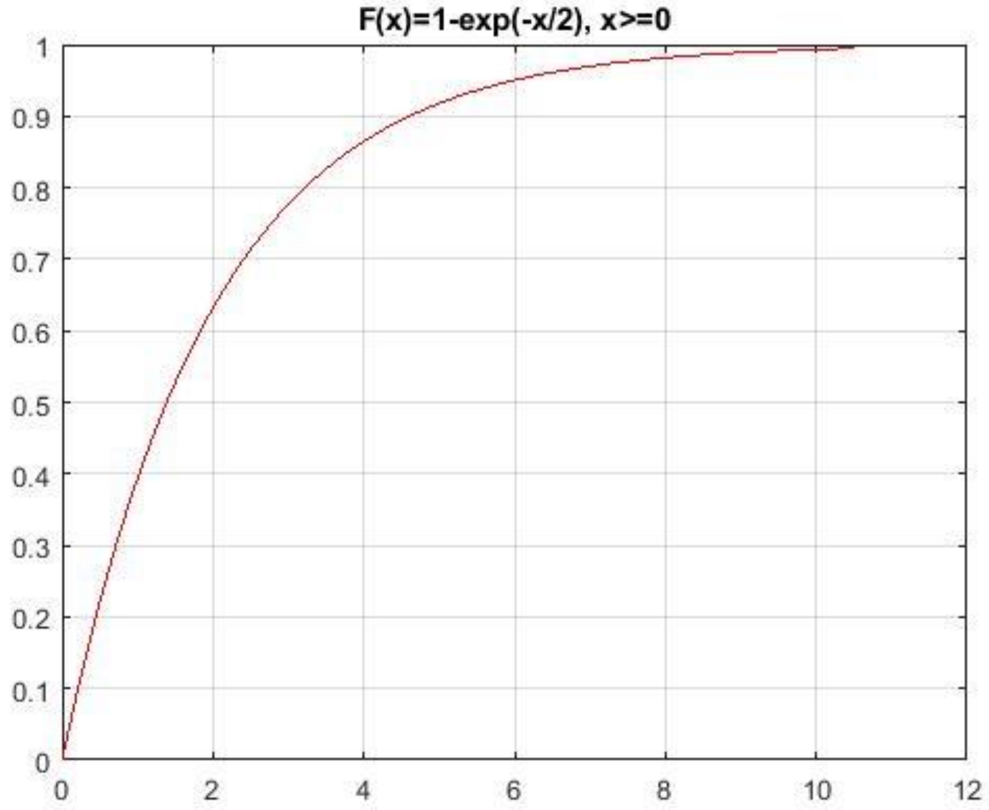
(c) If f is continuous, then $F'(x) = f(x)$.

(d) For $a \le b, P(a \le X \le b) = \int_{a}^{b} f(x) dx$.

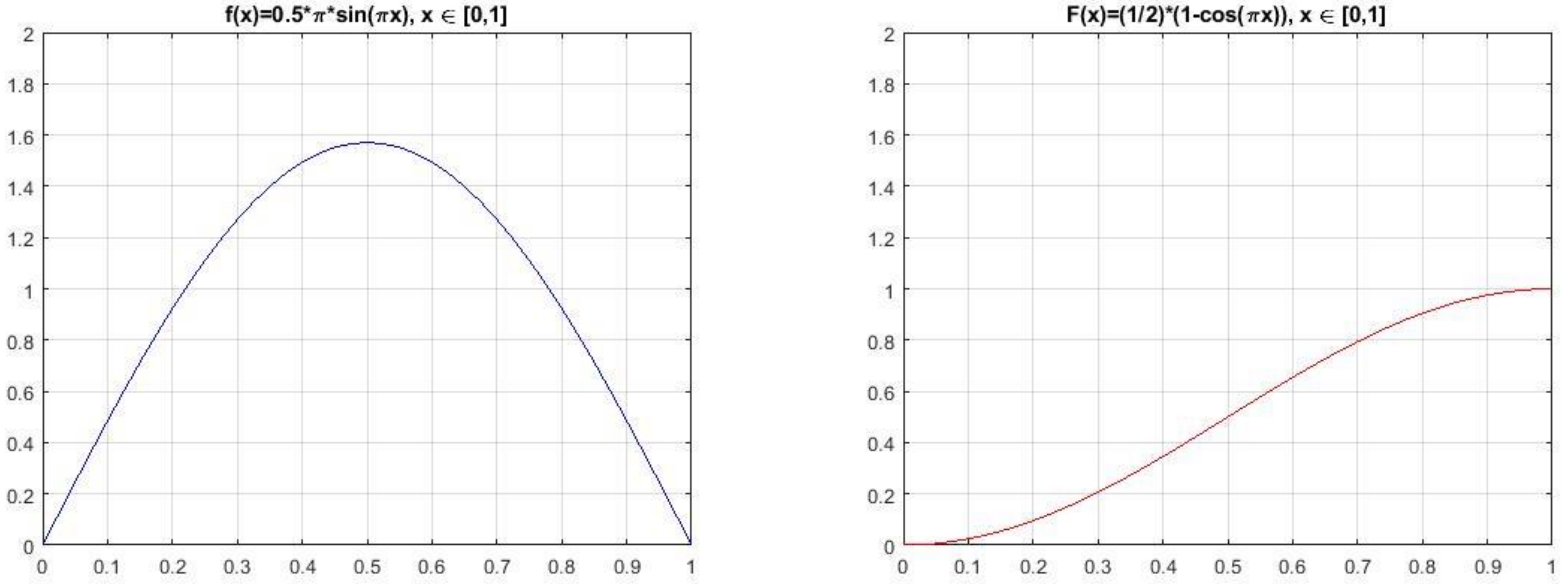
(e) $P(a < X < b) = P(a \le X < b) = P(a < X \le b) = P(a \le X \le b)$.

$$f(x) = \frac{1}{2}e^{-x/2} \text{ and } F(x) = 1 - e^{-x/2} \text{ (pdf vs. cdf)}$$





$$f(x) = \frac{\pi}{2}\sin(\pi x) \text{ vs. } F(x) = \frac{1}{2}(1 - \cos(\pi x))$$



Example 6.1 on P.245

Example 6.1 Experience has shown that while walking in a certain park, the time X, in minutes, between seeing two people smoking has a probability density function of the form

- *f(x)*=$\begin{cases} \lambda x e^{-x} & x > 0 \\ 0 \quad , & otherwise \end{cases}$

(a) Calculate the value λ. Ans: $\lambda = 1.$

(b) Find the distribution function of X. $F(t) = 1 - (t+1)e^{-t}\ if\ t \geq 0$.

(c) What is the probability that Jeff, who has just seen a person smoking, will see another person smoking in 2 to 5 minutes? in at least 7 minutes? P(2<X<5)$\approx 0.37, P(X \geq 7) \approx 8e^{-7} \approx 0.007$.

Example 6.2 on P.246

(6.2)(a) Sketch the graph of the function

$$f(x) = \begin{cases} \frac{1}{2} - \frac{1}{4}|x-3|, & 1 \leq x \leq 5 \\ 0 & otherwise \end{cases}$$

and show that it is the probability density function of a r.v. X.

(b) Find *F*, the distribution function of X, and sketch its graph.

(c*) Show that *F* is continuous.

Remark 6.1 on P.249.

Exercise 3 on P.250

3. The time it takes for a student to finish an aptitude test (in hours) has a probability density function of the form

$$f(x) = c(x-1)(2-x) \quad if\ 1 < x < 2$$

(a) Determine the constant c. Ans: $\int_1^2 f(x)dx = 1,\ so\ c = 6.$

(b) Calculate the distribution function of the time it takes for a randomly selected student to finish the aptitude test.

$$\text{Ans: } F(x) = \int_1^x f(t)dt = -2x^3 + 9x^2 - 12x + 5\ \ if\ 1 < x < 2,$$
$$and\ F(x) = 1\ if\ x \geq 2.$$

(a) What is the probability that a student will finish the aptitude test in less that 75 minutes (i.e., 1.25 hours)? Between 1.5 and 2 hours?

2. $F(x) = 1 - \frac{16}{x^2}\ if\ x \geq 4, and\ 0\ otherwise.$ Then

(a) $f(x) = \frac{32}{x^3}\ for\ x \geq 4\ \ (b)\ Sketch\ F\ and\ f\ \ for\ x \geq 4.$

$(c) P(5 < X < 7) = F(7) - F(5)$, (d) $P(X \geq 6) = 1 - F(6) = \frac{16}{36} = \frac{4}{9}$

Density Function of A Function of A R.V. (P.253)

In probability applications, we may face that the probability density function f of X is known but the p.d.f. h(X) is need. One of the solutions is to find the distribution function G of Y=h(X) first and compute the p.d.f of h(X) by g(y)=G'(y).

Example (6.3) Let X be a continuous r.v. with the probability density function

$$f(x) = \begin{cases} \frac{2}{x^2}, & if\ 1 < x < 2 \\ 0, & otherwise. \end{cases}$$

The distribution function and p.d.f. of $Y = X^2$ can be calculated as

$$G(y) = P(Y \le y) = P(X^2 \le y) = P\left(1 < X \le \sqrt{y}\right) = \int_1^{\sqrt{y}} \frac{2}{x^2}\, dx = 2 - \frac{2}{\sqrt{y}}\ ,$$

$$g(y) = G'(y) = \frac{1}{y\sqrt{y}} \quad for\ 1 < y < 4$$

Examples 6.4, 6.5 on P.254

(6.4) Let *X* be a continuous r.v. with p.d.f. *f*. In terms of *f*, *find the distribution and probability density functions of* $Y = X^3$.

Hint: $G(t) = P(Y \leq t) = P(X^3 \leq t) = P\left(X \leq \sqrt[3]{t}\right) = F(\sqrt[3]{t})$,

$$g(t) = G'(t).$$

(6.5) The error X of a measurement has the probability density function

$$f(x) = \begin{cases} \frac{1}{2} & if \ -1 < x < 1 \\ 0 & otherwise \end{cases}$$

Find the distribution and probability density functions of Y=|X|.

Theorem 6.1 Method of Transformations P.255

Let X be a continuous random variable with probability density function f_X and the set of possible values A. For the invertible function

$h: A \to R$, let $Y = h(X)$ be a random variable with the set of possible values *B*=*h*(*A*)=$\{h(a): a \in A\}$. Suppose that the inverse of $y = h(x)$ is the function $x = h^{-1}(y)$, which is *differentiable* for all values of $y \in B$. Then f_Y, the probability density function of *Y, is given by*

$$f_Y(y) = f_X\left(h^{-1}(y)\right)|(h^{-1})'(y)|, \ y \in B.$$

See the Proof on P.255

Hint: Compute $F(y) = P(Y = h(X) \leq y), then\ f_Y(y) = F'(y)$.

Example 6.6 on P.256

6.6 Let X be a random variable with the probability density function

$$f_X(x) = \begin{cases} 2e^{-2x} & if\ x > 0 \\ 0 & otherwise \end{cases}$$

The probability density function of $Y = \sqrt{X}$ is $\mathrm{f(y)} = 4\mathrm{y}e^{-2y^2}, y > 0$

Hint: $A = (0, \infty)$, $B = h(A) = (0, \infty)$

6.7 Let X be a random variable with the probability function

$$f_X(x) = \begin{cases} 4x^3 & if\ 0 < x < 1 \\ 0 & otherwise \end{cases}$$

The probability density function of $Y = 1 - 3X^2$ is $\mathrm{f(y)} = \frac{2}{9}(1 - y), y \in (-2,1)$

Hint: A=(0,1), and B=h(A)=(-2,1).

6.3 Expectations and Variances on P.259-264

Definition 6.2 If X is a continuous random variable with probability density function *f*, the expected value of X is defined as

$E(X) = \int_{-\infty}^{\infty} xf(x)dx$. μ or E[X] is an alternative notation.

Definition 6.3 If X is a continuous random variable with $E(X) = \mu$, then Var(X) and σ_X, called variance and standard deviation of X, respectively, are defined by

$\text{Var}(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$. $\sigma_X = \sqrt{Var(X)}$.

Examples of Expectation and Variance

(6.8) In a group of adult males, the difference between uric acid and 6, the standard value, is a random variable X with the following p.d.f.

$$f(x) = \frac{27}{490}(3x^2 - 2x) \quad if\ \frac{2}{3} < x < 3, \qquad f(x) = 0 \quad otherwise.$$

Find the mean of these differences for the group, that is, E(X). Ans: 2.36

(6.9) A random variable X with the following p.d.f. is called a Cauchy r.v.

$$f(x) = \frac{c}{1+x^2}, \quad -\infty < x < \infty$$

(a) Find c. c=$1/\pi$.

(b) Show that E(X) does not exist.

Theorems 6.2, 6.3, Corollary, P.261-264 (Skip)

Theorem 6.2 $E(X) = \int_0^\infty [1 - F(t)]\, dt - \int_0^\infty F(-t)dt$

Remark 6.4

Theorem 6.3 $E\big(h(X)\big) = \int_{-\infty}^{\infty} h(x)f(x)\, dx$

Example 6.10 A point X is selected from $(0,\pi/4)$ randomly. Calculate E(cos2X)=$\frac{2}{\pi}$, and E(cos^2X)=$\frac{1}{2}+\frac{1}{\pi}$.

Remarks 6.5, 6.6., 6.7: About $E(X^{n+1})$

$E(X^n)$: The $nth\ Moment\ of\ X$

- Let X be a discrete r.v. with probability mass function $f(x)$, then
- $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\, dx$ is called the nth moment of X if it exists.
- When $n = 1, E(X)\ is\ the\ expected\ value\ or\ expectation$.
- When $n = 2, E(X^2)\ is\ called\ the\ second\ moment$.
- Note that $Var(X) = E(X^2) - (E[X])^2$
- Remark: $E(X^{n+1})$ exits implies that $E(X^n)$. See proof on p.184.

A6. Let X be a discrete r.v. with the set of possible values $\{x_1, x_2, \cdots, x_n\}$; X is called uniform if $p(X = x_i) = \frac{1}{n},\ 1 \le i \le n$.

E(X)=$(n+1)/2$ and Var(X)=$(n^2-1)/12$ for the case that $x_i = i, 1 \le i \le n$.

Exercises A1,A6,A7,A10, B13,B14 on P.267-270

(A1) $F(x) = 1 - 16/x^2$ if $x \geq 4$, $F(x) = 0 \ if \ x < 4$. Then

(a) E(X)=8, (b) Show that V(X) does not exist.

(A6) $f(x) = 3e^{-3x} \ if \ 0 \leq x < \infty$. Then $E(e^X) = \frac{3}{2}$.

(A7) $f(x) = \frac{1}{\pi\sqrt{1-x^2}} \quad if \ -1 < x < 1$. Then E(X)=0.

(A10) $f(x) = \frac{2}{x^2} \ if \ 1 < x < 2$. Then $\mathrm{E(lnX)} = 1 - ln2$.

(B13) $f(x) = \frac{1}{\pi(1+x^2)}, \ if \ -\infty < x < \infty$.

Prove that $E(|X|^\alpha)$ converges if $0 < \alpha < 1$ and diverges if $\alpha \geq 1$.

(B14) $P(X > t) = \alpha e^{-\lambda t} + \beta e^{-\mu t}, \ t \geq 0, \alpha + \beta = 1$. Then E(X)= $\frac{\alpha}{\lambda} + \frac{\beta}{\mu}$.