Partial Solutions

1. Let $A, B \in R^{n\times n}$ be $upper - \Delta$. Show that $C = AB$ is also $upper - \Delta$.

Proof: Denote $A = [a_{ik}]$, $B = [b_{kj}]$, and $C = [c_{ij}]$, where $1 \le i, j, k \le n$. Since A, B are upper triangular, then $a_{ik} = 0\ for\ i > k$ and $b_{kj} = 0\ for\ k > j$.

By definition $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, we can write c_{ij} as follows

$$c_{ij} = \sum_{k=1}^{i-1} a_{ik}b_{kj} + \sum_{k=i}^{n} a_{ik}b_{kj}$$

Because A and B are both $upper - \Delta$, for $i > j$, we have $a_{ik} = 0$ since $i > k \ge 1$ in the first term of the above summation and $b_{kj} = 0$ since $k \ge i > j$ in the second term of the above summation. Thus, $c_{ij} = 0$ for $1 \le j < i \le n$. Hence, C is also $upper - \Delta$.

4. Gaussian elimination algorithm for solving a tridiagonal linear system.

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for i=1:n-1
  d(i+1)=d(i+1)-[(b(i+1)/d(i))*c(i)];
  f(i+1)=f(i+1)-[(b(i+1)/d(i))*f(i)];
end

x(n)=f(n)/d(n);
for k=n-1:-1:1
  x(k)=(f(k)-c(k)*x(k+1))/d(k);
end
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5.

(1) Since $y(x) = xe^x$, then $y' = y'(x) = e^x + xe^x$ and $y" = y"(x) = 2e^x + xe^x$, thus $-y" + (1 + x)y = -2e^x - xe^x + (1 + x)xe^x = -2e^x + x^2e^x$ by simple operations. Moreover, $y(0) = 0 \cdot e^0 = 0$ and $y(1) = 1 \cdot e^1 = e$.

(2) Let $z(x) = y_1(x) - y_2(x)$, where $y_1(x)$ and $y_2(x)$ are two solutions, we want to prove that $y_1(x) = y_2(x)$ by showing that $z(x) = 0\ \forall\ x \in [0, 1]$. Since $z(0) = y_1(0) - y_2(0) = 0 - 0 = 0$ and $z_(1) = y_1(1) - y_2(1) = e - e = 0$ and $z" = (y_1)" - (y_2)" = \cdots = (1 + x)(y_1 - y_2) = (1 + x)z$, hence $-z" + (1 + x)z = 0$.